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Near-Exact Analytical Solution of the SIR-Model for the Precise Temporal Dynamics of Epidemics

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Abstract: A near-exact analytical solution of the statistical Susceptible-Infectious-Recovered (SIR) epidemics model for a constant ratio k_0 of infection to recovery rates is derived. The derived solution is not of inverse form as the known solutions in the literature but expresses rather directly the three compartmental fractions $S(\tau)$, $I(\tau)$ and $R(\tau)$ and thus the rate of new infections $j(\tau) = S(\tau)I(\tau)$ in terms of the single function $U(\tau)$ and the reduced time τ (the time-integrated infection rate), involving the principal and non-principal branches of Lambert's function. Exact analytical formulas for the peak time and the maximum fraction of $I(\tau)$ are obtained proving that the rate of new infections peaks before the fraction of infected persons. Our analysis is not entirely analytically exact because the reduced time dependence of the function $U(\tau)$ obeying a nonlinear integro-differential equation is only obtained approximately by expanding a double-exponential function to first-order at small reduced times, and employing an accurate simple approximation of the principal Lambert function at large times, respectively.

Keywords: coronavirus; statistical analysis; COVID-19; pandemic spreading

1. Introduction

The Susceptible-Infectious-Recovered (SIR) model is the simplest of the compartmental models used for the mathematical modeling of infectious diseases in order to reproduce or predict the temporal evolution of infectious diseases in human populations. Originally developed nearly hundred years ago [1, 2] it lately has become very popular and widespread [3–43] due to its successful applications to the outbreaks of the corona virus in many countries [44]. The considered population of $N \gg 1$ initially susceptible persons is assigned to the three compartments $S(t)$ (susceptible), $I(t)$ (infectious), or $R(t)$ (recovered/removed). Persons from the population may progress between these compartments described by the time-dependent infection ($a(t)$ and recovery ($\mu(t)$) rates.

The three respective population fractions obey the sum constraint condition

$$I(t) + S(t) + R(t) = 1 \tag{1}$$

at any time, and their temporal evolution is given by the SIR-equations

$$\frac{dS(t)}{dt} = -a(t)S(t)I(t), \tag{2a}$$

$$\frac{dI(t)}{dt} = a(t)S(t)I(t) - \mu(t)I(t), \tag{2b}$$

$$\frac{dR(t)}{dt} = \mu(t)I(t). \tag{2c}$$

Besides numerical solutions it is of high interest to derive analytical solutions of the underlying dynamical SIR-equations. In the most general case of a time-dependent ratio $k(t) = \mu(t)/a(t)$ between the recovery and the infection rate analytical approximate solutions were derived [45, 46] which are very accurate if the cumulative fraction of infections $J(t) = 1 - S(t)$ is small compared to unity at all times.

Recently, new analytical solutions became available [47, 48] for an arbitrary time dependence of the infection and recovery rates, provided that the ratio between the two rates is independent of time, for two different types of initial conditions. We refer to these in the following as KSSIR solutions. The utility of the KSSIR solutions were proven by their successful application to past waves of the corona virus [47]. However, in both cases the KSSIR solutions could only be given in inverse form $t(S)$ involving an integral, that had to be approximated by second-order polynomials (see [47] for details). Here we consider an alternative approach to the KSSIR solution that avoids the inverse form adopting the semi-time initial conditions [49–59]

$$S(t_0) = 1 - \eta, \qquad I(t_0) = \eta, \qquad R(t_0) = 0, \tag{3}$$

with $0 < \eta \simeq \mathcal{O}(1/N) \ll 1$ denoting the initial seed infection fraction of the population.

Two quantities are of particular interest in studies of infections [60–101]:

(1) the differential rate of newly infected persons from the desease

$$\dot{J}(t) = a(t)I(t)S(t), \tag{4}$$

which, with a delay time t_d of about a week, determines the death rate $d(t) = fNj(t - t_d)$, where the mortality rate f is of the order 10^{-2}–10^{-3} varying for different mutants of the Covid virus and different countries [102]. $\dot{J}(t)$ also determines the hospitalization rate of seriously infected persons.

(2) the fraction of infected persons $I(t)$ determines the peak time of required clinical resources in the host country of the considered population [103].

Both quantities $\dot{J}(t)$ and $I(t)$ first increase in time, undergo a maximum and drop at late times. While exact analytical formulas for the peak time τ_j and the peak rate of new infections $\dot{J}_{\max}$ are available in the KSSIR-model, several different approximations for the peak time τ_j and the peak fraction of infected persons $I_{\max}$ have been derived [103]. It is one purpose of the present study to derive exact expressions for τ_I and $I_{\max}$.

2. Reduction of the General SIR-Equations

By introducing the reduced time

$$\tau = \int_{t_0}^{t} d\xi\, a(\xi) \tag{5}$$

for arbitrary but given real time dependent infection rates $a(t)$ and the ratio

$$k(\tau(t)) = \frac{\mu(t)}{a(t)} \tag{6}$$

the SIR-Equations (1)–(3) can be written as

$$\frac{dS(\tau)}{d\tau} = -S(\tau)I(\tau), \tag{7a}$$

$$\frac{dI(\tau)}{d\tau} = S(\tau)I(\tau) - k(\tau)I(t), \tag{7b}$$

$$\frac{dR(\tau)}{d\tau} = k(\tau)I(\tau), \tag{7c}$$

$$1 = S(\tau) + I(\tau) + R(\tau), \tag{7d}$$

subject to initial conditions

$$S(\tau = 0) = 1 - \eta, \quad I(\tau = 0) = \eta, \quad R(\tau = 0) = 0. \tag{8}$$

From the invariant $\dot{J}(t)dt = j(\tau)d\tau$ we obtain with Equation (5) in the form $d\tau/dt = a(t)$ for the differential rate of newly infected persons from the desease

$$j(\tau) = \frac{dJ(\tau)}{d\tau} = \frac{\dot{J}(t)}{a(t)} = S(\tau)I(\tau) = -\frac{dS(\tau)}{d\tau} \tag{9}$$

with the initial value $j(0) = \eta(1-\eta)$. The invariant ($J(t) = J(\tau)$) cumulative distribution corresponding to $j(\tau)$ is given by

$$J(\tau) = \int_{-\infty}^{\tau} d\tau' j(\tau') = J(0) - \int_0^{\tau} d\tau' \frac{dS(\tau')}{d\tau'} = J(0) + S(0) - S(\tau) = 1 - S(\tau), \tag{10}$$

where the initial condition $J(0) = I(0) = \eta$ had been used, in accord with Equation (8).

2.1. Reduction

Equation (7a) readily yields

$$I(\tau) \quad = \quad -\frac{d\ln S(\tau)}{d\tau} = -\frac{dS(\tau)/d\tau}{S(\tau)}, \tag{11}$$

whereas Equation (7b) provides

$$\frac{d\ln I(\tau)}{d\tau} \quad = \quad S(\tau) - k(\tau), \tag{12}$$

which with the initial condition on $I(0) = \eta$ integrates to

$$I(\tau) \quad = \quad \eta e^{U(\tau)}, \tag{13}$$

$$U(\tau) \quad = \quad \int_0^{\tau} dx\,[S(x) - k(x)]. \tag{14}$$

Combining Equations (11) and (13) with

$$S(\tau) \quad = \quad \frac{dU(\tau)}{d\tau} + k(\tau) \tag{15}$$

then leads to the single nonlinear differential equation

$$\frac{d}{d\tau}\ln\left[\frac{dU(\tau)}{d\tau} + k(\tau)\right] \quad = \quad -\eta e^{U(\tau)} \tag{16}$$

for $U(\tau)$. Equation (16) integrates to

$$\frac{dU(\tau)}{d\tau} + k(\tau) = (1-\eta)\exp\left[-\eta\int_0^{\tau} dx\, e^{U(x)}\right], \tag{17}$$

where we made use of the initial condition $S(0) = 1 - \eta$.

2.2. Stationary Ratio

Throughout this study a stationary ratio (6) is assumed, i.e.,

$$k(\tau) = k_0 = \text{const.} \tag{18}$$

k_0 often is referred to as inverse reproduction number. The derived exact analytical solutions then hold for stationary infection and recovery rates as well as for any time-dependent infection rate $a(t)$ provided the recovery rate $\mu(t) \propto a(t)$ has the same time variation while its absolute value can be different.

For a stationary ratio Equations (14) and (16)–(17) simplify to

$$U(\tau) = \int_0^\tau dx\, S(x) - k_0\tau, \tag{19a}$$

$$\frac{d}{d\tau}\ln\left[\frac{dU(\tau)}{d\tau}+k_0\right] = -\eta e^{U(\tau)}, \tag{19b}$$

$$\frac{dU(\tau)}{d\tau}+k_0 = (1-\eta)e^{-\eta\int_0^\tau dx\, e^{U(x)}}, \tag{19c}$$

$$\frac{d^2U(\tau)}{d\tau^2} = -\eta(1-\eta)U(\tau)e^{-\eta\int_0^\tau dx\, e^{U(x)}}. \tag{19d}$$

In earlier work [48] Equations (7) were solved exactly in inverse form as

$$\tau = \int_\eta^J \frac{dx}{(1-x)\left(x+k_0\ln\frac{1-x}{1-\eta}\right)}, \tag{20}$$

allowing important conclusions on the final values of the fractions S, I, R (see Appendix A for details). Here we will follow a different approach avoiding the necessary inversion of solution (20) to derive $J(\tau)$. However, the noted exact results from Appendix A will be used below to check the validity of the alternative solution.

3. Exact Solution

3.1. Ansatz

The ansatz

$$\frac{dU(\tau)}{d\tau}+k_0 = -k_0 W\left(-ae^{be^{U(\tau)}}\right) \tag{21}$$

in terms of the Lambert [104] function $W(Z)$ (see Appendix G of ref. [105]) and yet unspecified constants a and b provides for Equation (19c)

$$-k_0W(-ae^{be^{U(\tau)}}) = (1-\eta)e^{-\eta\int_0^\tau dx\, e^{U(x)}}. \tag{22}$$

Note that the ansatz (21) is equivalent to

$$-k_0\tau = \int^U \frac{dx}{1+W(-ae^{be^x})}. \tag{23}$$

Next, we will determine the constants a and b and thus prove that the ansatz (21) fulfills the differential Equation (19c). Applying the defining equation for the Lambert function

$$W(Z) = Ze^{-W(Z)} \tag{24}$$

yields for Equation (22)

$$\eta\int_0^\tau dx\, e^{U(x)} = W\left(-ae^{bU(\tau)}\right) - be^{U(\tau)} - \ln\left(\frac{k_0 a}{1-\eta}\right). \tag{25}$$

Hence for its derivative with respect to τ, with

$$Z(\tau) = -ae^{be^{U(\tau)}}, \tag{26}$$

one obtains

$$\begin{aligned}
\eta e^{U(\tau)} &= \frac{dU}{d\tau}\left[\frac{d}{dU}W(-ae^{be^U}) - be^U\right] = \frac{dU}{d\tau}\left[\frac{dZ}{dU}\frac{d}{dZ}W(Z) - be^U\right] \\
&= \frac{dU}{d\tau}\left[be^U Z\frac{d}{dZ}W(Z) - be^U\right] = \frac{dU}{d\tau}be^U\left[Z\frac{d}{dZ}W(Z) - 1\right] \\
&= \frac{dU}{d\tau}be^U\left[\frac{W(Z)}{1+W(Z)} - 1\right] = -\frac{dU}{d\tau}\frac{be^{U(\tau)}}{1+W(Z)},
\end{aligned} \tag{27}$$

where we used the differential equation

$$Z\frac{dW(Z)}{dZ} = \frac{W(Z)}{1+W(Z)} \tag{28}$$

for Lambert functions. The Equation (27) can be written as

$$-\frac{d\tau}{dU} = \frac{b/\eta}{1+W(Z)}, \tag{29}$$

which is solved by

$$-\tau = \frac{b}{\eta}\int^{U} \frac{dx}{1+W(-ae^{be^x})}. \tag{30}$$

In order for the solution (30) to be consistent with Equation (23) one has to demand that $b = \eta/k_0$. Consequently, the exact solution is given by

$$-k_0\tau = \int^{U} \frac{dx}{1+W\left(-ae^{\frac{\eta e^x}{k_0}}\right)}, \tag{31}$$

where we still have to determine the constant a and the lower integration limit from the initial condition and we have to identify the appropriate branch of the Lambert function.

If we re-use $U(0) = 0$ and $U'(0) = 1-\eta-k_0$, then we can determine a from Equation (27) evaluated at $\tau = 0$,

$$1 = -\frac{1-\eta-k_0}{k_0[1+W(-ae^{\eta/k_0})]}, \tag{32}$$

or equivalently

$$W(-ae^{\eta/k_0}) = -\frac{1-\eta}{k_0}. \tag{33}$$

With Equation (24) written as

$$Z = We^{W}, \tag{34}$$

Equation (33) can be solved for a and provides

$$a = \frac{(1-\eta)e^{-1/k_0}}{k_0} = -\alpha, \tag{35}$$

with α from Equation (A9) in Appendix A.

To conclude, Equation (21) is formally solved by

$$\tau = -\frac{1}{k_0}\int^{U} \frac{dx}{1+W\left(-\frac{1-\eta}{k_0}e^{-(1-\eta e^x)/k_0}\right)} = -\frac{1}{k_0}\int^{U} \frac{dx}{1+W\left(\alpha e^{\frac{\eta e^x}{k_0}}\right)}, \tag{36}$$

but we have to make this more precise, as the Lambert function W has two branches, W_0 and W_{-1}.

For positive k_0 and $\eta \in [0,1)$ the argument of the Lambert function is negative because $\alpha < 0$. Recall that $W(Z) = -1$ for $Z = -1/e$, where the two Lambert branches meet. Let τ_U and $U_{\max}$ denote the peak time and peak amplitude of $U(\tau)$. The peak amplitude $U_{\max} = U(\tau_U)$ is determined by the solution of $Z = -1/e$, and thus given by

$$U_{\max} = \ln\left[-\frac{1+\ln(a)}{b}\right] = \ln\left[\frac{1-k_0-k_0\ln\frac{1-\eta}{k_0}}{\eta}\right]. \tag{37}$$

For times up to peak time the solution (36) applies, using the non-principal branch of Lambert's function,

$$\tau = -\frac{1}{k_0}\int_0^{U} \frac{dx}{1+W_{-1}\left(\alpha e^{\frac{\eta e^x}{k_0}}\right)} \qquad (0 \le \tau \le \tau_U), \tag{38}$$

with the peak time τ_U determined by

$$\tau_U = -\frac{1}{k_0}\int_0^{U_{\max}} \frac{dx}{1+W_{-1}\left(\alpha e^{\frac{\eta e^x}{k_0}}\right)}. \tag{39}$$

Over this interval $U(\tau)$ monotonically increases from $U(0) = 0$ to $U(\tau_U) = U_{\max}$. Beyond peak time, the solution is determined by

$$\tau = \tau_U + \frac{1}{k_0} \int_U^{U_{\max}} \frac{dx}{1 + W_0\left(\alpha e^{\frac{\eta e^x}{k_0}}\right)} \qquad (\tau \geq \tau_U). \tag{40}$$

Over this remaining interval $U(\tau)$ monotonically decreases towards $\lim_{\tau\to\infty} U(\tau) = -\infty$.

The respective slopes below and above the peak time are obtained by taking the derivative with respect to τ of Equations (38) and (40) providing

$$\frac{dU}{d\tau}(\tau \leq \tau_U) = -k_0 \left[1 + W_{-1}\left(\alpha e^{\frac{\eta e^{U(\tau)}}{k_0}}\right)\right] \tag{41}$$

and

$$\frac{dU}{d\tau}(\tau \geq \tau_U) = -k_0 \left[1 + W_0\left(\alpha e^{\frac{\eta e^{U(\tau)}}{k_0}}\right)\right], \tag{42}$$

respectively. At very large times the latter approaches

$$\lim_{\tau\to\infty} \frac{dU}{d\tau} = -k_0[1 + W_0(\alpha)], \tag{43}$$

since $\lim_{\tau\to\infty} U(\tau) = -\infty$.

3.2. Resulting Fractions

Using the notation $W_s(Z)$ and $W_s'(Z) = dW_s(Z)/dZ$ with $s = -1$ for $\tau \leq \tau_U$ and $s = 0$ for $\tau \geq \tau_U$, respectively, and

$$Z = -\frac{1-\eta}{k_0} e^{\frac{\eta e^{U(\tau)} - 1}{k_0}} = \alpha e^{\frac{\eta e^{U(\tau)}}{k_0}} = -\exp\left[\frac{\eta}{k_0}\left(e^{U(\tau)} - e^{U_{\max}}\right) - 1\right], \tag{44}$$

with negative α, the solutions Equations (41) and (42) read

$$\frac{dU}{d\tau} = -k_0[1 + W_s(Z)]. \tag{45}$$

Then, according to Equation (15) one finds

$$S(\tau) = U'(\tau) + k_0 = -k_0 W_s(Z) = -k_0 W_s\left(\alpha e^{\frac{\eta e^{U(\tau)}}{k_0}}\right), \tag{46}$$

so that with $Z_\infty = \alpha$

$$S_\infty = S(\tau = \infty) = -k_0 W_0(\alpha), \tag{47}$$

in agreement with the exact KSSIR result (A10a). Using

$$\frac{dZ}{d\tau} = \frac{\eta}{k_0} \frac{dU(\tau)}{d\tau} e^{U(\tau)} Z = -\eta Z e^{U(\tau)} \left[1 + W_s(Z)\right], \tag{48}$$

one obtains for the first derivative of Equation (46) with respect to τ

$$\frac{dS}{d\tau} = -k_0 \frac{dZ}{d\tau} W_s'(Z) = \eta k_0 Z e^{U(\tau)} [1 + W_s(Z)] W_s'(Z) = \eta k_0 e^{U(\tau)} W_s(Z), \tag{49}$$

where we used Lambert's equation (28). Inserting Equations (46) and (49) yields for Equation (11)

$$I(\tau) = -\frac{dS(\tau)/d\tau}{S(\tau)} = \eta e^{U(\tau)} = -\frac{d}{d\tau} \ln\left[U'(\tau) + k_0\right] = -\frac{U''(\tau)}{U'(\tau) + k_0}, \tag{50}$$

where we used Equation (19b), thus correctly reproducing the earlier Equation (13). Obviously, the fraction of infected persons peaks at τ_U, because of its dependence $\propto e^{U(\tau)}$, and its maximum value is given by

$$I_{\max} = \eta e^{U_{\max}} = 1 - k_0 - k_0 \ln \frac{1-\eta}{k_0} \tag{51}$$

Consequently, one finds for the rate of new infections (9)

$$j(\tau) = S(\tau)I(\tau) = -U''(\tau) = \eta(1-\eta)U(\tau)e^{-\eta\int_0^\tau dx\, e^{U(x)}}, \tag{52}$$

where in the last step we used Equation (19d), and the corresponding cumulative fraction of infected persons

$$J(\tau) = 1 - S(\tau) = 1 + k_0 W_s(Z) = 1 + k_0 W_s\left(\alpha e^{\frac{\eta e^{U(\tau)}}{k_0}}\right). \tag{53}$$

Likewise, the sum constraint (7d) leads to

$$R(\tau) = 1 - S(\tau) - \frac{j(\tau)}{1 - J(\tau)} = 1 - k_0 - U'(\tau) + \frac{U''(\tau)}{U'(\tau) + k_0} \tag{54}$$

As $U(\infty) = -\infty$ we derive

$$I_\infty = j_\infty = 0, \tag{55a}$$

$$R_\infty = J_\infty = 1 - S_\infty = 1 + k_0 W_0(\alpha), \tag{55b}$$

reproducing exactly the earlier noted properties (A8) and (A10).

We thus have expressed all quantities of interest, the fractions S, I, R as well as the differential rate of new infections and its corresponding cumulative number in terms of the function $U(\tau)$ and its first and second derivatives. These expressions are exact. It remains to derive the direct reduced time dependence of the function $U(\tau)$ which is done approximately for large and small times in the following sections.

4. Approximated $U(\tau)$ for Large and Small Times

4.1. Large Times $\tau \geq \tau_U$

We note that the function $Z(\tau)$ in Equation (44) has values

$$-Z(\tau_U) = e^{-1}, \qquad -Z(\tau = \infty) = -\alpha = \frac{(1-\eta)e^{-\frac{1}{k_0}}}{k_0}, \tag{56}$$

so that $0 < |Z(\tau = \infty)| < |Z(\tau_U)| < e^{-1}$. For such small values of Z we then use as approximation

$$W_0(Z) \simeq (1 + eZ)^{1/2} - 1, \tag{57}$$

shown in Figure 1 in comparison to the exact variation. As can be seen the agreement is sufficient, and the approximation exact at the terminals.

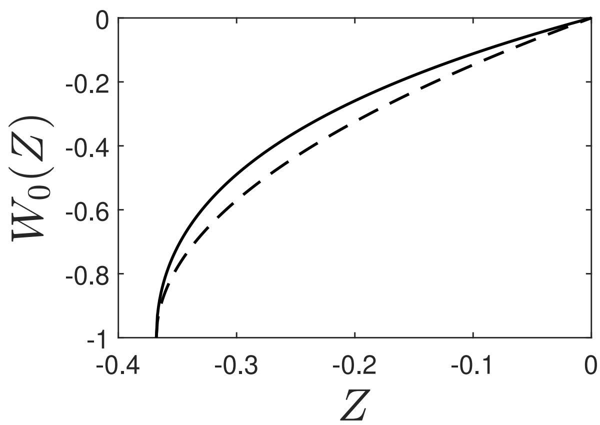

Figure 1. Principal branch of Lambert's function. Approximation (57) (dashed) for $W_0(Z)$ (solid).

The approximation (57) then yields for Equation (40)

$$k_0(\tau - \tau_U) \simeq \int_U^{U_{\max}} \frac{dx}{\sqrt{1 + \alpha e \exp(\eta e^x/k_0)}} = \int_{\frac{\eta e^U}{k_0}}^{\frac{\eta e^{U_{\max}}}{k_0}} \frac{dy}{y\sqrt{1 + \alpha e e^y}}, \tag{58}$$

where we substituted $y = \eta e^x / k_0$. The upper integration limit is given by

$$\mathcal{O}(k_0) = \frac{\eta e^{U_{\max}}}{k_0} = \frac{1}{k_0} - \left[1 + \ln \frac{1-\eta}{k_0}\right] \simeq \frac{1}{k_0} + \ln(k_0) - 1, \tag{59}$$

where the second approximation holds for small values of $\eta \ll 1$. It is shown in Figure 2 as a function of k_0. The upper integration limit is thus smaller than unity provided $k_0(2 - \ln k_0) > 1$, corresponding to

$$k_0 > -W_{-1}(-e^{-2}) \approx 0.32 \tag{60}$$

in agreement with Figure 2.

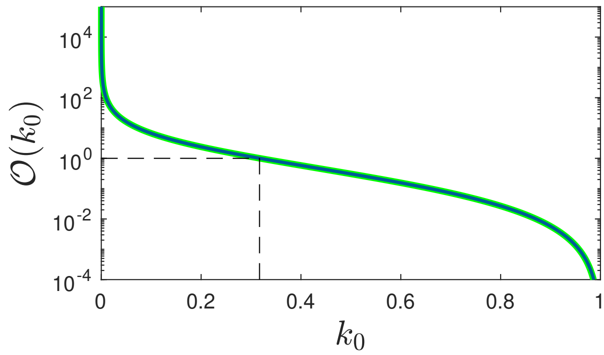

Figure 2. Upper bound $\mathcal{O}(k_0) = \eta e^{U_{\max}}/k_0$ versus k_0 for $\eta = 10^{-5}$. Shown is both the exact expression (black line) and the approximant (59) (green line). The dashed lines mark $\mathcal{O}(k_0) = 1$. The curve is basically unaltered for smaller η.

For such values of k_0 we approximate

$$1 + \alpha e e^y \simeq 1 + \alpha e + \alpha e y \tag{61}$$

to obtain with

$$\kappa = -\frac{\alpha e}{1 + \alpha e} = \left[\frac{k_0}{1-\eta} e^{\frac{1}{k_0} - 1} - 1\right]^{-1} = \left[e^{\frac{\eta e^{U_{\max}}}{k_0}} - 1\right]^{-1} = \frac{1}{e^{\mathcal{O}(k_0)} - 1} \tag{62}$$

for the integral (58)

$$\sqrt{1 + \alpha e}\, k_0(\tau - \tau_U) = \frac{k_0(\tau - \tau_U)}{\sqrt{1+\kappa}} \simeq \int_{\frac{\eta e^U}{k_0}}^{\frac{\eta e^{U_{\max}}}{k_0}} \frac{dy}{y\sqrt{1 - \kappa y}} \tag{63}$$

With the substitution $s = 1 - \kappa y$ one finds for the last equation

$$\begin{aligned} \frac{k_0(\tau - \tau_U)}{\sqrt{1+\kappa}} &= \int_{1 - \frac{\eta\kappa e^{U_{\max}}}{k_0}}^{1 - \frac{\eta\kappa}{k_0} e^U} \frac{ds}{(1-s)\sqrt{s}} = 2\left[\tanh^{-1}\sqrt{s}\right]_{1 - \frac{\eta\kappa e^{U_{\max}}}{k_0}}^{1 - \frac{\eta\kappa}{k_0} e^U} \\ &= \left[\ln \frac{1 + \sqrt{s}}{1 - \sqrt{s}}\right]_{1 - \frac{\eta\kappa e^{U_{\max}}}{k_0}}^{1 - \frac{\eta\kappa}{k_0} e^U} \end{aligned} \tag{64}$$

After straightforward algebra Equation (64) leads to

$$\frac{\eta\kappa e^{U(\tau)}}{k_0} = 1 - \tanh^2\left[\frac{k_0(\tau - \tau_U)}{2\sqrt{1+\kappa}} + \Phi\right] = \cosh^{-2}\left[\frac{k_0(\tau - \tau_U)}{2\sqrt{1+\kappa}} + \Phi\right] \tag{65}$$

with the constant Φ defined by

$$\Phi = \frac{1}{2} \ln \frac{1 + \sqrt{1 - \frac{\eta\kappa e^{U_{\max}}}{k_0}}}{1 - \sqrt{1 - \frac{\eta\kappa e^{U_{\max}}}{k_0}}} = \tanh^{-1}\sqrt{1 - \frac{\kappa\eta}{k_0} e^{U_{\max}}} \tag{66}$$

Equation (65) readily provides as approximation at large times

$$U(\tau \geq \tau_U) = U_H(\tau) = \ln \frac{k_0}{\eta\kappa} - 2\ln\cosh\zeta(\tau), \tag{67}$$

$$\zeta(\tau) = \frac{k_0(\tau-\tau_U)}{2\sqrt{1+\kappa}} + \Phi. \tag{68}$$

We note that Equation (67) correctly provides $U_H(\tau_U) = U_{\max}$ as $\cosh(\zeta(\tau_U)) = \cosh\Phi = \sqrt{k_0/\eta\kappa e^{U_{\max}}}$.

The slope of the approximation (67) is

$$U'(\tau \geq \tau_U) = -\frac{k_0}{\sqrt{1+\kappa}} \tanh\zeta(\tau), \tag{69}$$

providing for its limiting slope

$$\lim_{\tau\to\infty} \frac{dU_H}{d\tau} = -\frac{k_0}{\sqrt{1+\kappa}} = \sqrt{k_0\left[k_0 - (1-\eta)e^{1-\frac{1}{k_0}}\right]}, \tag{70}$$

which Figure 3 compares favorably well with the exact limiting slope given by Equation (43).

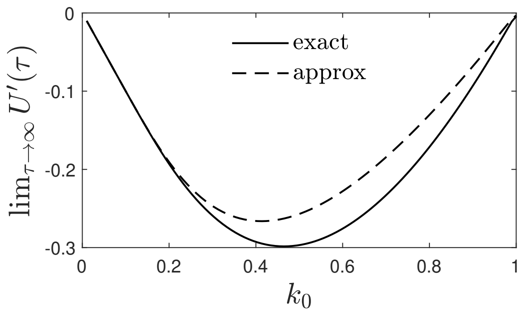

Figure 3. Limiting slope of $U(\tau)$. Exact numerical result using Equation (43) and approximation (70). Following [106] the numerical solution of the GSL equations we obtained using the 10th order predictor–corrector Adams method [107, 108]. Within 0.1% precision, a single-step solver based on a modified Rosenbrock formula of order 2, implemented by [109] as ode23s in Matlab$^{\mathrm{TM}}$ yielded practically indistinguishable results.

It is tempting to use approximation (67) to calculate the corresponding $U'(\tau \geq \tau_U)$ as in Equation (69) and $U''(\tau \geq \tau_U)$ to infer directly the three fractions S, I and R as well as the differential rate j at large times. However, this produces incorrect results as can be seen with the resulting

$$S(\tau \geq \tau_U) = U'(\tau \geq \tau_U) + k_0 = k_0[1 - \frac{\tanh\zeta(\tau)}{\sqrt{1+\kappa}}] \tag{71}$$

implying

$$S_\infty = k_0\left(1 - \frac{1}{\sqrt{1+\kappa}}\right) = k_0\left(1 - \sqrt{1 - \frac{1-\eta}{k_0}e^{1-\frac{1}{k_0}}}\right) \simeq \frac{1-\eta}{2k_0}e^{1-\frac{1}{k_0}} \tag{72}$$

which is finite but slightly disagrees with the exact final value (47), as shown in Figure 4.

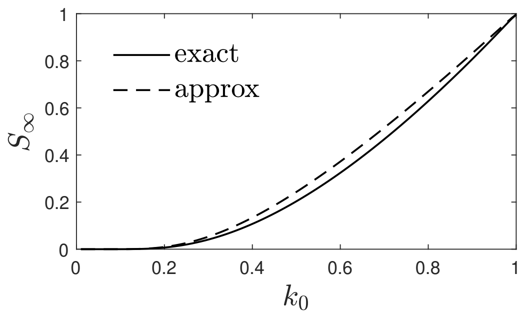

Figure 4. Limiting susceptible fraction. Exact (47) and approximate (72) analytic expressions for S_∞. For this plot, $\eta = 10^{-5}$, but the situation is very comparable for any $\eta \ll 1$.

The proper way to continue is to only use Equation (67) as an approximation for $U(\tau \geq \tau_U)$ and to insert it in the earlier general expressions for the fractions. With this approximation we obtain for Equation (44)

$$Z(\tau \geq \tau_U) = \alpha \exp\left(\frac{1}{\kappa \cosh^2 \zeta(\tau)}\right), \tag{73a}$$

$$Z(\tau_U) = \alpha \exp\left(\frac{\eta e^{U_{\max}}}{k_0}\right) = -\frac{1}{e}, \tag{73b}$$

and consequently for the fraction (46)

$$\begin{aligned} S(\tau \geq \tau_U) &= 1 - J(\tau \geq \tau_U) = -k_0 W_0(Z(\tau \geq \tau_U)) \\ &= -k_0 W_0 \left[\alpha \exp\left(\frac{1}{\kappa \cosh^2 \zeta(\tau)}\right)\right], \end{aligned} \tag{74}$$

which in contrast to the incorrect Equation (72) now correctly approaches $S_\infty = -k_0 W_0(\alpha)$. For later use we note

$$S(\tau_U) = -k_0 W_0(-e^{-1}) = k_0 \tag{75}$$

Likewise, the fraction (50) at large times is given by

$$I(\tau \geq \tau_U) = k_0 \ln \frac{Z}{\alpha} = \frac{k_0}{\kappa \cosh^2 \zeta(\tau)}, \tag{76}$$

reproducing correctly $I_\infty = 0$. The rate of new infections (9) then is

$$\begin{aligned} j(\tau \geq \tau_U) &= S(\tau \geq \tau_U) I(\tau \geq \tau_U) = -\frac{k_0^2 W_0 \left[\alpha \exp\left(\frac{1}{\kappa \cosh^2 \zeta(\tau)}\right)\right]}{\kappa \cosh^2 \zeta(\tau)} \\ &= -k_0^2 W_0(Z) \ln \frac{Z}{\alpha}. \end{aligned} \tag{77}$$

4.2. Small Times $\tau \leq \tau_U$

For small times where $U(\tau) \ll 1$ we expand the double-exponential function on the right-hand side of Equation (19c) to first order as

$$\begin{aligned} e^{-\eta \int_0^\tau dx\, e^{U(x)}} &\simeq 1 - \eta \int_0^\tau dx\, e^{U(x)} \simeq 1 - \eta \int_0^\tau dx\, [1 + U(x)] \\ &= 1 - \eta\tau - \eta \int_0^\tau dx\, U(x), \end{aligned} \tag{78}$$

so that Equation (19c) becomes

$$\frac{dU(\tau)}{d\tau} + \eta(1-\eta)\left[\tau + \int_0^\tau dx\, U(x)\right] \simeq 1 - \eta - k_0, \tag{79}$$

fulfilling the correct initial condition $U'(0) = 1 - \eta - k_0$. Setting

$$\int_0^\tau dx\, U(x) = F(\tau) - \tau + \frac{1 - \eta - k_0}{\eta(1-\eta)}, \tag{80}$$

Equation (79) reduces to

$$\frac{d^2 F(\tau)}{d\tau^2} + \eta(1-\eta) F(\tau) = 0, \tag{81}$$

with the solution

$$F(\tau) = C_1 \sin(\rho\tau) + C_2 \cos(\rho\tau), \qquad \rho = \sqrt{\eta(1-\eta)}. \tag{82}$$

Therefore

$$\begin{aligned} \int_0^\tau dx\, U(x) &= C_1 \sin(\rho\tau) + C_2 \cos(\rho\tau) - \tau + \frac{1 - \eta - k_0}{\eta(1-\eta)}, \\ U(\tau) &= \rho[C_1 \cos(\rho\tau) - C_2 \sin(\rho\tau)] - 1. \end{aligned} \tag{83}$$

The two integration constants C_1 and C_2 are determined by the conditions $U(0) = 0$ and $U(\tau_U) = U_{\max}$ yielding

$$C_1 = \frac{1}{\rho}, \qquad C_2 = \frac{\cos(\rho\tau_U) - 1 - U_{\max}}{\rho \sin(\rho\tau_U)}. \tag{84}$$

Consequently

$$U(\tau \le \tau_U) \;=\; U_L(\tau) = g(\tau) - 1, \tag{85a}$$
$$g(\tau) \;=\; \frac{\sin\rho(\tau_U - \tau) + (1 + U_{\max})\sin(\rho\tau)}{\sin(\rho\tau_U)}. \tag{85b}$$

Since $U(\tau_U) = U_{\max}$ guarantees, according to Equations (44) and (46), that $Z(\tau_U) = -e^{-1}$ implying $W_{-1}(Z(\tau_U)) = -1$, with the approximation (85) also $S(\tau_U) = k_0$ is in agreement with Equation (75).

For general small times Equation (44) subjected to the approximation Equation (85a) provides

$$Z(\tau \le \tau_U) = \alpha e^{\frac{\eta}{k_0} e^{g(\tau)-1}}, \tag{86}$$

so that Equation (46) leads to

$$S(\tau \le \tau_U) = 1 - J(\tau \le \tau_U) = -k_0 W_{-1}\left(\alpha e^{\frac{\eta}{k_0} e^{g(\tau)-1}}\right). \tag{87}$$

Likewise, the fraction (50) at small times is given by

$$I(\tau \le \tau_U) = \eta e^{g(\tau)-1}, \tag{88}$$

reproducing correctly $I(\tau = 0) = \eta$. The rate of new infections (9) then is

$$j(\tau \le \tau_U) = S(\tau \le \tau_U) I(\tau \le \tau_U) = -\eta k_0 e^{g(\tau)-1} W_{-1}\left(\alpha e^{\frac{\eta}{k_0} e^{g(\tau)-1}}\right). \tag{89}$$

We recall that Equation (39) determines

$$\tau_U \;=\; -\frac{1}{k_0}\int_0^{U_{\max}} \frac{dx}{1 + W_{-1}\left(\alpha e^{\frac{\eta e^x}{k_0}}\right)}. \tag{90}$$

so that for given values η and k_0 all parameters are fixed.

In most applications the initial fraction of infected persons $\eta \simeq \mathcal{O}(10^{-5})$ is very small. Hence for reduced times $\tau \le \tau_U \ll \eta^{-1/2}$ one can further approximate

$$\sin(\rho\tau) \simeq \rho\tau, \qquad \sin(\rho\tau_U) \simeq \rho\tau_U, \qquad \sin[\rho(\tau_U - \tau)] \simeq \rho(\tau_U - \tau) \tag{91}$$

to obtain for the function (85b)

$$g(\tau \le \tau_U) \;\simeq\; 1 + \frac{\tau}{\tau_U} U_{\max}, \tag{92}$$

i.e., one may replace $g(\tau) - 1$ in Equations (86)–(89) by $\tau U_{\max}/\tau_U$ with $U_{\max}$ from Equation (37).

5. Results

5.1. Rate of New Infections

According to Equations (77) and (89) with our earlier notation the rate of new infections at all reduced times is given by

$$j(\tau) \;=\; -k_0^2 W_s(Z_s) \ln \frac{Z_s}{\alpha} \tag{93}$$

with

$$Z_{-1} = Z_L \;=\; \alpha e^{\frac{\eta}{k_0} e^{g(\tau)-1}}, \qquad \tau \le \tau_U \tag{94a}$$
$$Z_0 = Z_H \;=\; \alpha e^{\kappa^{-1} \cosh^{-2} \zeta(\tau)}, \quad \tau \ge \tau_U \tag{94b}$$

We note that

$$\frac{Z_L}{\alpha} \in [e^{\frac{\eta}{k_0}}, e^{\mathcal{O}(k_0)}] \geq 1, \tag{95a}$$

$$\frac{Z_H}{\alpha} \in [1, e^{\mathcal{O}(k_0)}] > 1. \tag{95b}$$

In the last column of Figure 5 we compare this rate of new infections based on τ_U from Equation (90) with the exact numerical solution for $\eta = 10^{-5}$ and several choices of the parameter $k_0 = 0.2, 0.5, 0.8$. One notices excellent agreement between the analytical and numerical curves in all three cases.

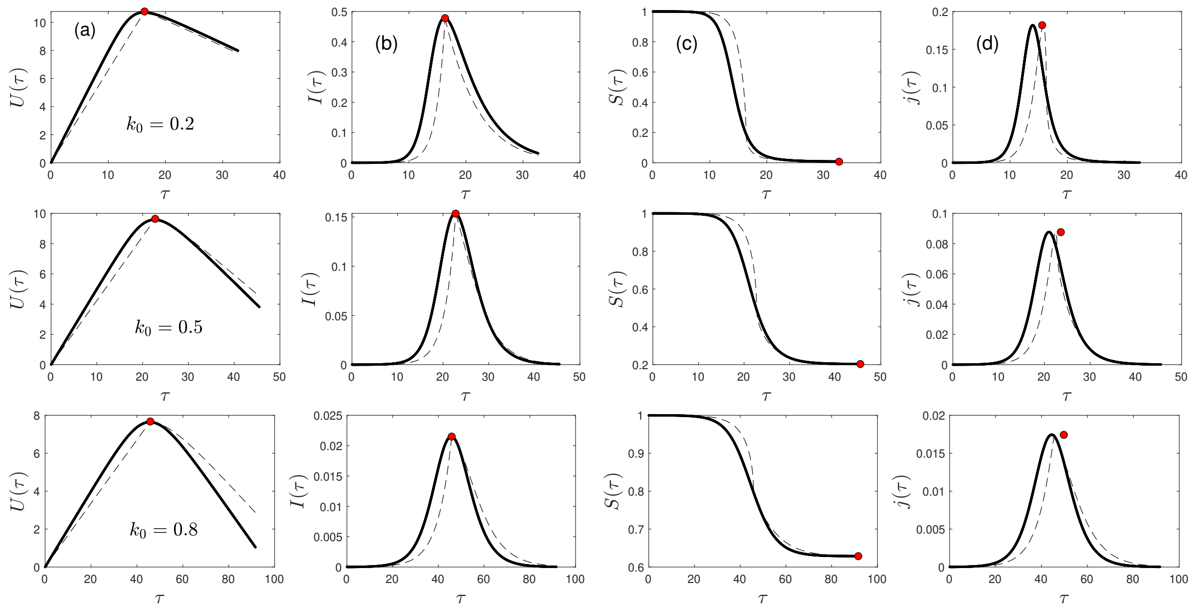

Figure 5. Time evolution of the functions $U(\tau)$, $I(\tau)$, $S(\tau)$, and $j(\tau)$. (**a**) Numerical $U(\tau)$ from (14) (solid) compared with $U(\tau)$ from (67) and (85) (dashed), using ζ from Equation (68), Φ from Equation (66), ρ from (82), κ from (62), τ_U from (39), $U_{\max}$ from (37). The red bullet marks $(\tau_U, U_{\max})$. (**b**) Numerical $I(\tau)$ (solid) compared with $I(\tau)$ from Equations (76) and (88). The red bullet marks $(\tau_I = \tau_U, I_{\max})$ from Equations (39) and (51). (**c**) Numerical $S(\tau)$ (solid) compared with $S(\tau)$ from Equations (87) and (74). The red bullet marks S_∞ (47). (**d**) Numerical $j(\tau)$ (solid) compared with $j(\tau)$ from Equations (93) with Z_s according to Equation (94). The red filled bullet marks $(\tau_j, j_{\max})$ according to Equations (103) and (106). Parameters: $\eta = 10^{-5}$ and $k_0 \in \{0.2, 0.5, 0.8\}$ mentioned in the left panel.

The rate of new infections (93) attains its maximum for a vanishing first derivative

$$\begin{aligned}
\frac{dj(\tau)}{d\tau} &= -k_0^2 \frac{dZ_s}{d\tau}\frac{d}{dZ_s}[W_s(Z_s)\ln\frac{Z_s}{\alpha}] \\
&= \eta k_0^2 Z_s e^{U_s}[1+W_s(Z_s]\left[W_s'(Z_s)\ln\frac{Z_s}{\alpha} + \frac{W_s(Z_s)}{Z_s}\right] \\
&= \eta k_0^2 e^{U_s} W_s(Z_s)[1+W_s(Z_s)]\left[1+\frac{Z_s W_s'(Z_s)}{W_0(Z_s)}\ln\frac{Z_s}{\alpha}\right] \\
&= \eta k_0^2 e^{U_s} W_s(Z_s)\left[1+W_s(Z_s)+\ln\frac{Z_s}{\alpha}\right] = 0,
\end{aligned} \tag{96}$$

where we used Equations (48) and (28). Thus the maximum occurs at Z_E given by the solution of

$$1 + W_s(Z_E) + \ln\frac{Z_E}{\alpha} = 0. \tag{97}$$

Taking the exponential of the last equation leads to

$$\frac{\alpha}{e} = Z_E e^{W_s(Z_E)} = W_s(Z_E) e^{2W_s(Z_E)}, \tag{98}$$

where we used Equation (24). Setting $X = 2W_s(Z_E)$ one can cast Equation (98) into the form

$$e^{-X} = \frac{e}{2\alpha}X \tag{99}$$

with the solution

$$X = W_s\left(\frac{2\alpha}{e}\right) \tag{100}$$

and consequently

$$W_s(Z_E) = \frac{1}{2}W_s(\alpha_0), \qquad \alpha_0 = \frac{2\alpha}{e}, \tag{101}$$

where we introduced α_0. Applying Equation (34) then provides

$$Z_E = \frac{1}{2}W_s(\alpha_0)e^{\frac{1}{2}W_s(\alpha_0)} = -\frac{1}{2}\sqrt{\alpha_0 W_s(\alpha_0)}, \tag{102}$$

where Equation (24) and $\sqrt{-a} = \imath\sqrt{a}$ for positive values $a > 0$ has been used. The maximum is then given by

$$\begin{aligned} j_{\max} &= k_0^2 W_s(Z_E)[1 + W_s(Z_E)] = \frac{k_0^2}{4}W_s(\alpha_0)[2 + W_s(\alpha_0)] \\ &= \frac{k_0^2}{4}\left\{[1 + W_s(\alpha_0)]^2 - 1\right\}, \end{aligned} \tag{103}$$

where we inserted Equation (101). Equation (103) agrees exactly with the well-known KSSIR expression (A14) only if $s = -1$, i.e., only if the non-principal branch of the Lambert functions $W_{-1}(\alpha_0)$ in the solution (102) is chosen. The second solution $Z_E = -0.5\sqrt{\alpha_0 W_s(\alpha_0)}$ involving the principal branch $W_0(\alpha_0)$ can be ruled out as it provides values of $Z_E/\alpha = Z_H/\alpha$ smaller than unity, as can be seen by the dashed curves in Figure 6 that reside clearly outside of the possible values of Z_H/α according to Equation (95b).

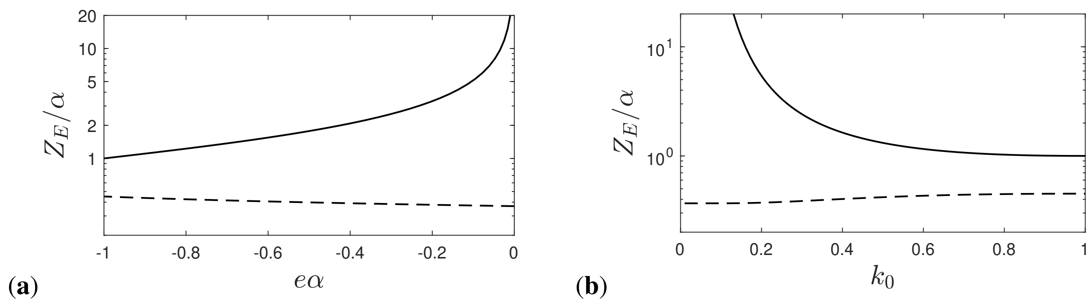

Figure 6. Extremum of $Z(\tau)$. Z_E/α (102) versus (**a**) $e\alpha$ and (**b**) k_0. Shown are the cases of $s = -1$ (solid) and $s = 0$ (dashed). For this plot $\eta = 10^{-5}$, but the plots are basically unaffected by η for $\eta \ll 1$.

In Figure 6 we calculate from Equation (102) Z_E/α as a function of k_0 for $\eta = 10^{-5}$. It can be seen that Z_E/α is always greater than unity. Because of the property (95a) this indicates that the peak time of the rate of new infections $\tau_j < \tau_U$ occurs at times smaller than τ_U and is given by the solution of the Eq.

$$e^{\frac{\eta}{k_0}e^{g(\tau_j)-1}} = \frac{W_{-1}(\alpha_0)}{2\alpha} = \frac{\alpha_0 e^{-W_{-1}(\alpha_0)}}{2\alpha} = e^{-[1+W_{-1}(\alpha_0)]}, \tag{104}$$

where we used Equation (24), so that

$$g(\tau_j) - 1 = \ln\left[\frac{k_0}{\eta}\ln\bigl(-[1 + W_{-1}(\alpha_0)]\bigr)\right] \tag{105}$$

With the approximation (92) one obtains

$$\tau_j = \frac{\tau_U}{U_{\max}}\ln\left[\frac{k_0}{\eta}\ln\bigl(-[1 + W_{-1}(\alpha_0)]\bigr)\right]. \tag{106}$$

In Figure 7 the ratio τ_j/τ_I is displayed as a function of k_0 for $\eta = 10^{-5}$. The ratio always is smaller than unity demonstrating that the rate of new infections peaks before the fraction of infected persons in agreement also with the second and fourth columns in Figure 5.

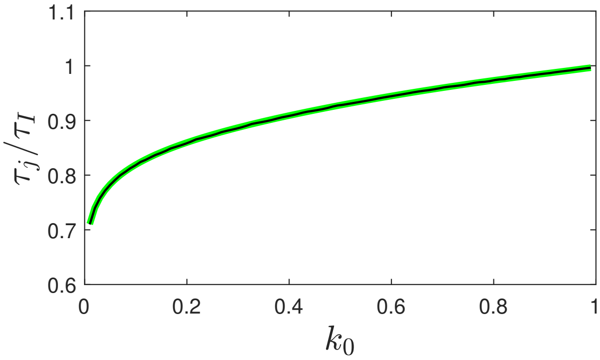

Figure 7. Ratio of peak times. Numerical $\tau_j/\tau_I = \tau_j/\tau_U$ (solid black) compared with this ratio using the analytical expression Equation (A15) with J_0 from Equation (A13) (thick green), which is well approximated by the simpler Equations (93) and (39) with Z_s according to Equation (94). For this figure, $\eta = 10^{-5}$.

5.2. Peak Time of Fraction of Infected Persons

The peak time τ_I of the fraction of infected persons is of particular interest [70, 75, 89, 110–120]. According to Equation (50) this peak time τ_I coincides with τ_U given exactly by Equation (39). Consequently, we can compare this exact peak time with approximants derived before [103]. Figure 8 demonstrates in the first column that the analytical equation (39) coincides with the numerically calculated peak time τ_I. While the earlier SK-I approximant (shown in the third column) provides acceptable agreement in a wide range of parameter values, the MT-approximant (shown in the second column) is less accurate.

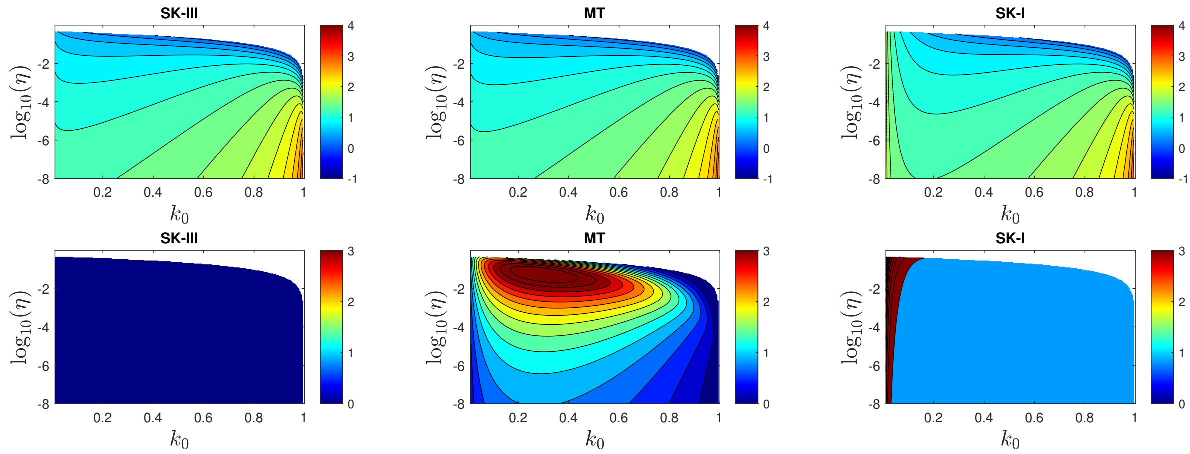

Figure 8. Comparison of analytical approximants for τ_I. The top row shows the decadic logarithm of τ_I (colormap) versus k_0 and η for $k_0 \leq 1 - 2\eta$, while the 2nd row shows the relative deviation (in %) between the exact τ_I and the approximant. The first column displays the analytical τ_I (39) (here denoted as SL-III), which coincides with the exact result, the second column the so-called MT-approximant proposed in Ref. [103], and the third column the SK-I-approximant [103].

6. Summary and Conclusions

We have derived a near-exact analytical solution of the statistical Susceptible-Infectious-Recovered (SIR) epidemics model for a constant ratio k_0 (referred to as KSSIR case) of infection ($a(t)$) to recovery ($\mu(t)$) rates in the semi-time case which is particularly appropriate for modeling the temporal evolution of later (than the first) pandemic waves when a greater population fraction from the first wave has been infected. By introducing the dimensionless reduced time variable $\tau = \int_{t_0}^{t} d\xi\, a(\xi)$ the derived solution holds for stationary rates as well as for the case of the same real time-dependency of the recovery and infection rates. The accuracy of the analytical solutions is confirmed by comparison with the exact numerical solutions of the SIR equations. Exact as well as accurately approximative solutions serve dual important purposes: first, they are suitable benchmarks for numerical codes, and secondly, they allow us to understand the fundamental behavior and functional patterns of epidemic outbursts as well as the decisive role of parameters.

The newly developed KSSIR-solution is not of inverse form as the known KSSIR solutions in the literature but rather directly expresses the three fractions $S(\tau)$, $I(\tau)$ and $R(\tau)$ and thus the rate of new infections $j(\tau) = S(\tau)I(\tau)$ exactly in terms of the same function $U(\tau) = -k_0\tau + \int_0^{\tau} dx\, S(x)$. With respect to the reduced time these fractions depend on two parameters: predominantly on the ratio k_0 and only weakly on the usually very small initial fraction η of infected persons. With respect to real time additionally the predescribed time dependent infection rate $a(t)$ enters via the reduced time. These exact expressions involve the principal and non-principal branches of the Lambert functions, which routinely are available in mathematical software packages such as Python (scipy), Excel, Matlab and Mathematica, above and below the peak time of the function $U(\tau)$ which agrees with the peak time of the rate of infections $I(\tau)$. The newly developed solution correctly reproduces all known exact expressions of the earlier KSSIR solution including the final values of S_∞, I_∞, and R_∞. It also provides exact analytical formulas for the peak time $\tau_I = \tau_U$ and the maximum fraction $I_{\max}$. These allow to check the accuracy of earlier derived approximants for τ_I. In particular it is shown that the rate of new infections peaks before the fraction of infected persons.

The derived near-exact solution is not entirely exact because the reduced time dependence of $U(\tau)$ obeying a nonlinear integro-differential equation is only obtained approximately for small and large times with respect to τ_U. At small reduced times where $U(\tau) \ll 1$ the approximation is based on the expansion of a double-exponential function to first-order, whereas at large reduced times an accurate simple approximation of the principal Lambert function $W_0(Z)$ is employed. The resulting rate of new infections correctly reproduces the known exact maximum rate of new infections.

Author Contributions

R.S.: conceptualization, methodology, formal analysis, writing-reviewing and editing; M.K.: methodology, formal analysis, software, writing-reviewing and editing, visualization. All authors have read and agreed to the published version of the manuscript.

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Informed Consent Statement

Not applicable.

Data Availability Statement

All data are enclosed with this publication.

Conflicts of Interest

The authors declare no conflict of interest.

Appendix A. Inverse KSSIR-Solution of Earlier Work

For stationary ratios (18) Equation (7c) immediately integrates to

$$R(\tau) = -k_0 \ln \frac{S(\tau)}{1-\eta} = -k_0 \ln \left[\frac{1-J(\tau)}{1-\eta}\right], \tag{A1}$$

where we used $S(\tau) = 1 - J(\tau)$ from Equation (10) so that also

$$I(\tau) = -\frac{d \ln[1-J(\tau)]}{d\tau} \tag{A2}$$

With these expressions the sum constraint (7c) then reads

$$J(\tau) + \frac{d \ln[1-J(\tau)]}{d\tau} + k_0 \ln \left[\frac{1-J(\tau)}{1-\eta}\right] = 0, \tag{A3}$$

implying

$$\frac{dJ(\tau)}{d\tau} = [1-J(\tau)] \left[J(\tau) + k_0 \ln \frac{1-J(\tau)}{1-\eta}\right] \tag{A4}$$

With the initial condition $J(0) = \eta$ Equation (A4) readily is solved in inverse form as

$$\tau = \int_{\eta}^{J} \frac{dx}{(1-x)[x + k_0 \ln \frac{1-x}{1-\eta}]}. \tag{A5}$$

This solution (A5) generalizes the known analytical solutions in the literature [1, 2, 121] as it holds for arbitrary time-dependence of the infection rate $a(t)$. The mentioned known solutions can be reproduced with Equation (A5) by setting $a_0(t - t_0)$ on its left-hand side resulting from a constant injection rate a_0.

Taking the derivative with respect to τ highlights the fact that the inverse integrand in (A5) is nothing but the differential rate of newly infected persons in terms of $J(\tau)$, i.e.,

$$j(\tau) \quad = \quad (1 - J(\tau)) \left[J(\tau) + k_0 \ln \frac{1 - J(\tau)}{1 - \eta} \right]. \tag{A6}$$

It has been noted before that important exact properties of the KSSIR-solution (A5) can be inferred without doing the inversion to $J(\tau)$.

Appendix A.1. Final And Maximum Values

The solution (A5) indicates that the maximum value $J_\infty = J(\tau = \infty)$ is attained when the denominator of the respective integrand vanishes, i.e.,

$$J_\infty + k_0 \ln \left(\frac{1 - J_\infty}{1 - \eta} \right) = 0. \tag{A7}$$

Consequently,

$$J_\infty = 1 + k_0 W_0(\alpha), \tag{A8}$$

where W_0 is the principal solution of Lambert's equation and

$$\alpha = -\frac{(1-\eta)e^{-1/k_0}}{k_0}. \tag{A9}$$

The knowledge of J_∞ from Equation (A8) immediately yields

$$S_\infty \quad = \quad 1 - J_\infty = -k_0 W_0(\alpha), \tag{A10a}$$

$$R_\infty \quad = \quad -k_0 \ln \frac{1 - J_\infty}{1 - \eta} = J_\infty, \tag{A10b}$$

$$I_\infty \quad = \quad 1 - S_\infty - R_\infty = 0, , \tag{A10c}$$

$$j_\infty \quad = \quad j(\tau = \infty) = 0. \tag{A10d}$$

Appendix A.2. Peak Differential Rate

Likewise, the maximum of the differential rate (A6) occurs when the derivative $(dj/dJ)_{J_0} = 0$ vanishes. With Equation (A6) one finds

$$\frac{dj}{dJ} = 1 - 2J - k_0[1 - \ln(1-\eta)] - k_0 \ln(1 - J), \tag{A11}$$

yielding for J_0 the transcendental equation

$$2J_0 = 1 - k_0 + k_0 \ln(1 - \eta) - k_0 \ln(1 - J_0), \tag{A12}$$

which is solved in terms of the non-principal Lambert function as

$$J_0 = 1 + \frac{k_0}{2} W_{-1}(\alpha_0), \qquad \alpha_0 = \frac{2\alpha}{e}, \tag{A13}$$

with α from (A9). Inserting Equation (A13) in Equation (A6) and making use of Equation (A12) yields for the maximum value in reduced time

$$j_{\max} = j(J_0) = (1 - J_0)(1 - J_0 - k_0) = \frac{k_0^2}{4} \left\{ [1 + W_{-1}(\alpha_0)]^2 - 1 \right\} \tag{A14}$$

According to Equation (A5) the peak time of the differential rate (A6) is given by

$$\tau_j = \int_{\eta}^{J_0} \frac{dx}{(1-x)[x + k_0 \ln \frac{1-x}{1-\eta}]}. \tag{A15}$$

For a maximum to occur at finite positive times $\tau_j > 0$, the derivative $dj/d\tau$ has to be positive at times $0 \le \tau < \tau_j$. With Equations (A11) and (A12) we readily find

$$\begin{aligned} \frac{dj}{d\tau} &= \frac{dJ}{d\tau}\frac{dj}{dJ} = j\frac{dj}{dJ} = j(\tau)[1 - 2J(\tau) - k_0(1 - \ln(1-\eta)) - k_0 \ln(1 - J(\tau))] \\ &= j(\tau)\left[2(J_0 - J(\tau)) - k_0 \ln \frac{1 - J(\tau)}{1 - J_0}\right]. \end{aligned} \tag{A16}$$

Since the requirement of a maximum $j_{\max}$ to exist at positive times is identical to the requirement of a positive $dj/d\tau$ at $\tau = 0$, we can insert $J(0) = \eta$ into the last equality in the first line of Eq. (A16) to find

$$\left.\frac{dj}{d\tau}\right|_{\tau=0} = \eta(1-\eta)[1 - 2\eta - k_0] > 0 \tag{A17}$$

implying

$$k_0 < 1 - 2\eta. \tag{A18}$$

For inverse reproduction numbers k_0 greater than $1 - 2\eta$, the daily rate is monotonically decreasing at all times from its initial value $j(0) = \eta(1-\eta)$ (decay phase). Contrary, for $k_0 < 1 - 2\eta$ the daily rate of newly infected persons attains a maximum at a finite positive time (peak case). At $k_0 = 1 - 2\eta$ the daily rate starts in its maximum at $\tau = 0$, and then decreases, while S, R and J approach their final values below unity.

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